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Efficient Finite Element and Differential Quadrature Methods for Heat Distribution in One-Dimensional Insulated-Tip Rectangular Fin

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Abstract. There are many numerical solution techniques acquainted by the computational mechanics community including the finite element method (FEM) and differential quadrature method (DQM). Usually elements are sub-divided uniformly in FEM (conventional FEM, CFEM) to obtain temperature distribution behavior in a fin. In CFEM extra computational complexity is needed to obtain a better solution with required accuracy. In this paper, non-uniform sub-elements techniques are considered for the FEM (efficient FEM, EFEM) solution to reduce the computational complexity. Then EFEM is applied for the solution of the one-dimensional heat transfer problem in an insulated-tip thin rectangular fin. The results are compared with CFEM and efficient DQM (EDQM, with non-uniform mesh generation). It is observed that EFEM exhibits more accurate results compared to CFEM and EDQM. The proposed techniques are showing the potentiality of the heat transfer related problem.

***Keywords*:** Heat transfer, Finite element method, Differential quadrature method, Rectangular fin.

1. Introduction

There are many numerical solution techniques well familiar by the computational mechanics community. FEM is one of those numerical solution techniques to solve structural, mechanical, heat transfer, and fluid dynamics which arise in problems of engineering and physical sciences [1-4]. The conventional FEM (CFEM) means the elements that we use are of same size and uniformly distributed. In its application to the solution of engineering problems, the finite element discretization has been implemented almost to the spatial problems. For dynamic or time dependent problems whose solutions as functions of time are of interest, a step by step procedure of finite difference is usually employed with computational complexity.

For heat transfer problems, rapid changes of heat/temperature distributions take place near the element boundary (and at the boundary). It is very important to know these temperature change behavior of an element prior to its use. Hence, to get an actual picture using FEM, the element is usually subdivided into very small sub-elements uniformly (conventional FEM, CFEM), which leads to huge amount of complexity, memory consumption and computational time (Park, 1996). Otherwise, error flow occurs with unreliable results [1-2,5].

On the other hand, to get a clear picture about the temperature changes near (and at) the element boundary, better to subdivide the elements into very small sub-elements at the boundary only, followed by relatively bigger elements gradually towards the mid-point of the element non-uniformly (efficient FEM, EFEM). This may serve the intended purpose without any additional burden and this is highlighted in this paper with improved accuracy (approximately 65%) compared to CFEM. Hence, here, focus is given to develop and apply efficient (non-uniform mesh density) nodal points distribution algorithm for automatic mesh (elements) generation to optimize CFEM solution.

DQM is another numerical solution technique to solve above mentioned problems efficiently [6-12]. The essence of the DQM is that the partial derivative of a function is approximated by a weighted linear sum of the function values at given discrete points. Bellman and Casti [6-7] developed this numerical solution technique in the early 1970s and since then, the technique has been successfully employed in a variety of problems in engineering and physical sciences. To make the DQM more efficient with less computational complexity, efficient DQM (EDQM) was proposed in[10-12] with non-uniformly distributed mesh points.

Hence, in this paper, one-dimensional (1-D) heat conduction problems in a thin insulated-tip rectangular fin is solved using EFEM by means of the accurate discretization and solver (code) and then the results are compared with CFEM and EDQM to verify EFEM efficiency. The paper is organized as follows. Section II presents the governing equation with EFEM rules, followed by simulation set-up and assumptions, results and discussions, and finally conclusion of the paper.

2. One-Dimensional Efficient Finite Element Method

One dimensional (1-D) heat conduction equation is shown in Eq. (1) [2,13-17].

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+Q=0 \tag{1}$$

with the boundary conditions $T\big|_{x=0}=T_0$ and $q\big|_{x=L}=h(T_L-T_\infty)$ as shown in figure 1. Here, the heat flux $q=-k\dfrac{dt}{dx}$.

Figure 1 shows the 1-D element discretization in the x-direction. The temperature T at various nodal points are the unknowns except at node 1, where, $T_1=T_0$ with initial temperature T_0. Within a typical element 'ei or e' the local node numbers are i and $i+1$ with coordinates x_i and x_{i+1} and element length, $l_{ei}=x_{i+1}-x_i$. For example, $e1$ whose local node numbers are 1 and 2 with coordinates x_1 and x_2, and element length $l_{e1}=x_2-x_1$ respectively.

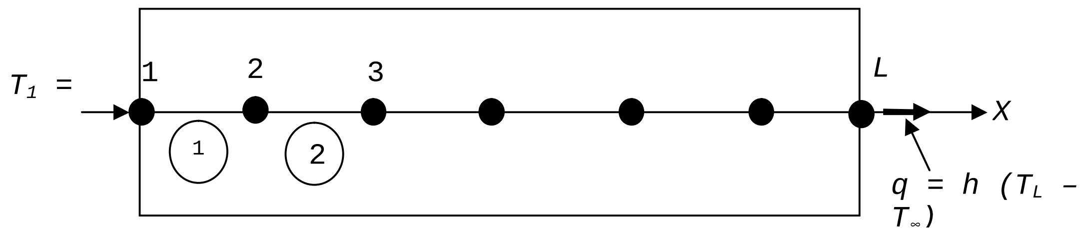

Figure 1. Boundary conditions for 1-D heat conduction

One-dimensional thin rectangular fin is shown in figure 2. Heat is transmitted along its length by conduction and the tip of the fin is insulated. The governing equation for the temperature in the fin is given in Eq. (1).

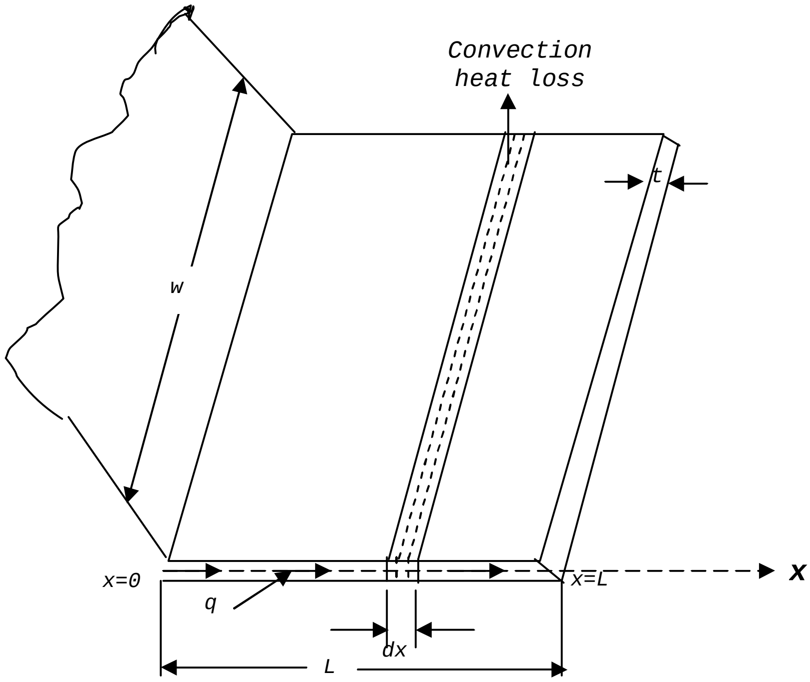

Figure 2. Thin rectangular fin

The parameter, M is given by $M^2 = \dfrac{hp}{kA_C}$,

where, p is the fin perimeter (m) and A_c is the cross sectional area of the fin [m^2]. Fin length, width and thickness are L, w and t respectively.

In this case, The heat flux is $q = h(T - T_\infty) = -k\dfrac{dT}{dx}$, perimeter, $p = 2(w + t)$, cross-section area, $A_C = w \times t$ and $\dfrac{p}{A_C} = \dfrac{2(w+t)}{w \times t} \approx \dfrac{2}{t}$.

The convection heat loss in the fin is equivalent to negative heat source and can be expressed as follows: $Q = -\dfrac{(p\,dx)h(T - T_\infty)}{A_C dx} = -\dfrac{ph}{A_C}(T - T_\infty)$.

After manipulating, Eq. (1) can be expressed in the form of Eq. (2).

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) - \frac{ph}{A_C}(T - T_\infty) = 0 \tag{2}$$

To calculate the approximate numerical solution *T(x)*, the mathematical formulation using Galerkin's approach [2] is written in Eq. (3).

$$\int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dT}{dx}-\frac{ph}{A_C}(T-T_\infty)\right)\right]dx=0 \tag{3}$$

where ϕ is a test function constructed from the same basis functions as those of *T*, with $\phi(0)=0$. Integrating by parts Eq. (3) becomes,

$$\phi k\frac{dT}{dx}\bigg|_0^L-\int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx-\int_0^L \phi\frac{ph}{A_C}(T-T_\infty)dx=0 \tag{4}$$

Since $\phi(0)=0$ and $q=h(T_L-T_\infty)$, Eq. (4) is expressed as in Eq. (5),

$$-\phi(L)h(T_L-T_\infty)-\int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx-\int_0^L \phi\frac{ph}{A_C}(T-T_\infty)dx=0 \tag{5}$$

A global virtual temperature vector is defined as $\psi=[\psi_1,\psi_2,\psi_3,\ldots,\psi_L]^T$ then within each element, the test function becomes, $\phi(i)=N_i\psi_i$. Here, N is the element shape function and $N_L=1$ at the element boundary (figure 1). Therefore we can write as Eq. (6).

$$\phi(L)=[N\psi]_L=\psi_L \tag{6}$$

As, $\frac{dT}{dx}=\mathbf{B}_T\mathbf{T}^e$, from Eq. (6), $\frac{d\phi}{dx}=\mathbf{B}_T\psi$, then, $\left(\frac{d\phi}{dx}\right)^T\times\left(\frac{dT}{dx}\right)=\left(\mathbf{B}_T^T\boldsymbol{\psi}^T\right)\left(\mathbf{B_T}\mathbf{T}^{ei}\right)$ and $\mathbf{B}_T^{\mathbf{T}}\mathbf{B}_T=\frac{1}{(l_{ei})^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$.

The element conductivity matrix is expressed in Eq. (7).

$$k_T=\frac{k_{ei}\,l_{ei}}{2}\int_{-1}^{1}\mathbf{B}_T^{\mathbf{T}}\mathbf{B}_T\,d\xi=\frac{k_{ei}}{l_{ei}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \tag{7}$$

where, ξ varies from -1 to $+1$ and $\xi=\frac{2}{x_{i+1}-x_i}(x-x_i)-1$ $\quad with\ d\xi=\frac{2}{x_{i+1}-x_i}dx$.

The element heat rate vector due to the heat source is written by Eq. (8).

$$\mathbf{R}=\mathbf{r}_Q=\frac{Q_{ei}\,l_{ei}}{2}\int_{-1}^{1}\mathbf{N}^{\mathbf{T}}d\xi=\frac{Q_{ei}l_{ei}}{2}\begin{bmatrix}1\\ 1\end{bmatrix} \tag{8}$$

With the help of Eqs. (3-8), Eq. (2) can be transformed into either Eq. (9) or Eq. (10)

$$-\psi_l h(T_L - T_\infty) - \sum_{ei} \boldsymbol{\psi}^{\mathbf{T}} \left(\frac{k_{ei}\, l_{ei}}{2} \int_{-1}^{1} \mathbf{B}_T^{\mathbf{T}} \mathbf{B}_T d\xi \right) \mathbf{T}^e + \sum_{ei} \boldsymbol{\psi}^{\mathbf{T}} \frac{Q_{ei}\, l_{ei}}{2} \int_{-1}^{1} \mathbf{N}^{\mathbf{T}} d\xi = 0 \tag{9}$$

Or,

$$\boldsymbol{\psi}^{\mathbf{T}} \mathbf{K}_T \mathbf{T} + \psi_L h T_L = \boldsymbol{\psi}^{\mathbf{T}} \mathbf{R} + \psi_L h T_\infty \tag{10}$$

For insulated-tip fin, the base of the fin is held at a constant temperature, T_0 and the tip of the fin is insulated, and the final global matrix shown in Eq. (10) can be written as Eq. (11).

$$\begin{pmatrix} A_{22} & A_{23} & \dots A_{2L} \\ A_{32} & A_{33} & \dots A_{3L} \\ \dots & \dots & \dots\ \dots \\ A_{L2} & A_{L3} & \dots A_{LL} \end{pmatrix} \begin{pmatrix} T_2 \\ T_3 \\ \dots \\ .T_L \end{pmatrix} = \begin{pmatrix} R_{2\infty} \\ R_{3\infty} \\ \dots \\ R_{L\infty} \end{pmatrix} - \begin{pmatrix} A_{21}T_0 \\ A_{31}T_0 \\ \dots \\ A_{L1}T_0 \end{pmatrix} \tag{11}$$

Using Eq. (11) and the efficient FEM (EFEM) algorithm, the approximate solution *T(x)* has been obtained. The 1-D EFEM algorithm (rule) is depicted in terms of self-explanatory flow chart in figure 3. Example of non-uniform and uniform sub-element distributions and their length calculations are depicted in figures 4 and 5 respectively.

Simulation Set-Up and Assumptions

Table 1 shows the parameters we consider and their corresponding values that are used to obtain simulation results using FORTRAN 90 software. We used these values to obtain the temperature distribution for EFEM, CFEM, EDQM and exact solution. We assumed $M^2 = hP/kA = 1$ and the associated assumptions (in Table I) to compare the obtained FEM results with DQM [11] and exact solution [17]. We need to remark that, to obtain 1-D DQM solutions, element material properties, fin-width and fin-thickness are not required (which is the shortcoming of the method). The errors in FEM and DQM solutions are computed and compared to the exact solutions.

Table 1. Input Parameters and Assumptions for 1-D Rectangular Fin

Input Parameters	Assumed value for *Insulated-Tip* Fin
Boundary and other values:	
Initial temperature (T0)	1 ^{O}C
Ambient temperature (T∞)	0 ^{O}C
Heat flux (q)	0 at x = 1
% Error threshold (eh)	0 - 0.1
Element Type (NNODE):	
Linear (for 1-D)	2
Element material properties:	
Thermal conductivity ($k_e = k$)	7.03125 *W/(m ^{0}C)*
Convective heat transfer coefficient *(h)*	9 *W/m^2 ^{0}C*
Heat source *(Q)*	0 *W/m^3 ^{0}C*
Element (Fin) dimension:	
length (*L*) along x-axis	1 m
width (*w*)	Variable to make *M* = 1
thickness (*t*)	Variable to make *M* = 1
Number of elements (*N*)	11 - 104

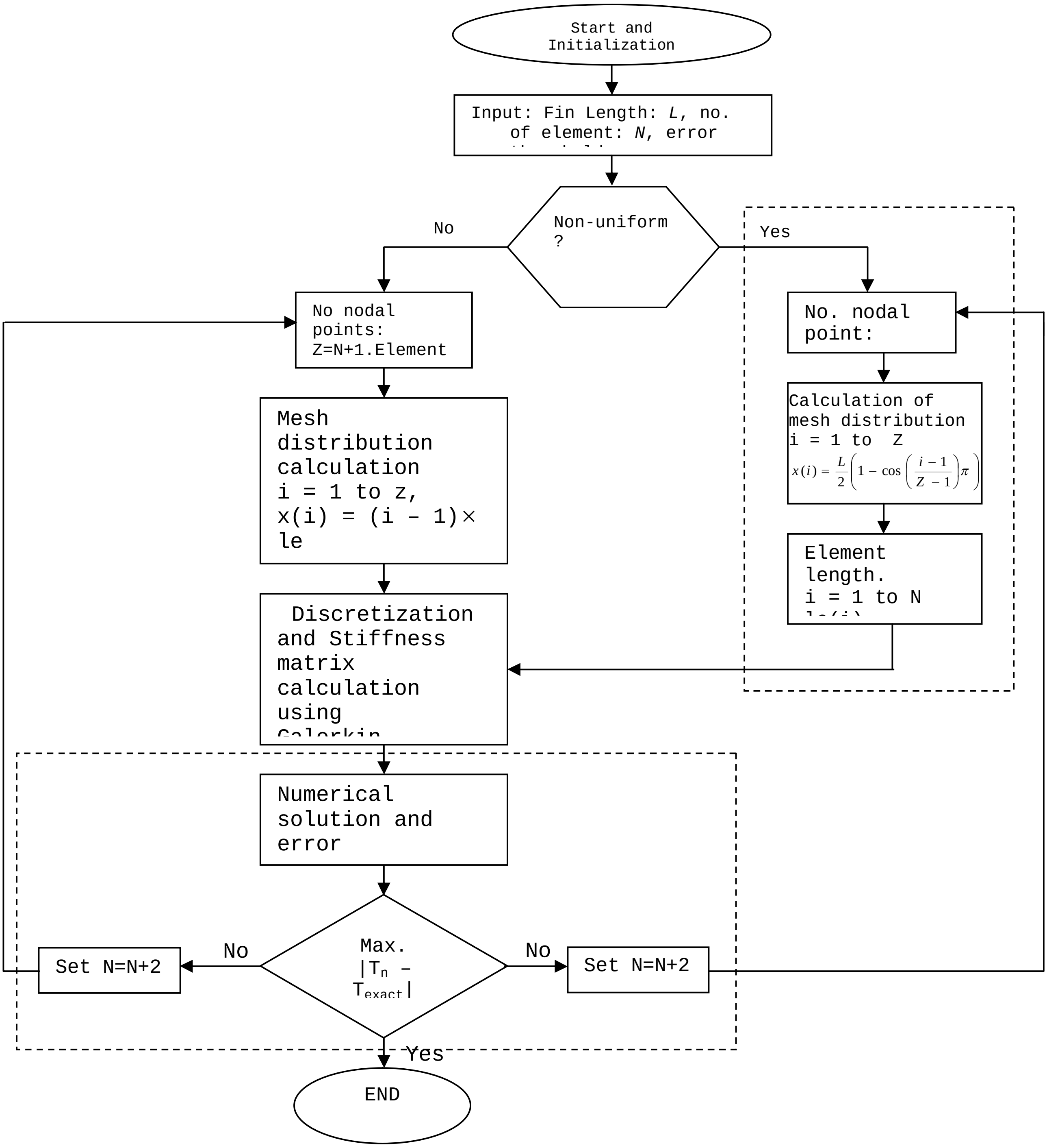

Figure 3. Efficient discretization and solution rule for 1-D FEM.

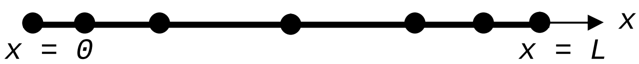

Figure 4. Example 1-D efficient (non-uniform) sub-element lengths distribution.

Figure 5. Example 1-D conventional (uniform) sub-element lengths distribution.

4. Results and Discussion

The convergence of the FEM solution for conventional and optimum mesh distribution have been compared and shown in figures 6 and 7 (magnified version). Both uniform FEM and non-uniform FEM solutions in terms of maximum % errors show a monotonic convergence with the increasing number of mesh points (shown up to $N = 104$). It is apparent that non-uniform FEM results show similar (or sometimes less) accuracy for N $\leq$ 11, but yields result with higher accuracy, of one order of magnitude or more with increasing N (for N $\geq$ 30) compared to that with uniform FEM. The results converge rapidly up to N = 101 and at this point the best results are obtained, whereas uniform FEM results converge slowly throughout the solution domain and then diverge for N $\geq$ 900 (as tested) without showing the best results like non-uniform FEM. It happens due to the mesh point distribution strategy of equally spaced and unequally spaced nodal points in the computational domain and the inherited complexity to compute the stiffness matrix for equally spaced nodal points. This shows that for FEM, the solution for non-uniform mesh FEM is better than those of uniform mesh FEM.

Comparisons of convergence of fin-temperature in terms of maximum % error for FEM versus DQM solution are shown in figures 8 to 10. It is apparent that the uniform DQM solutions converge up to N = 41, oscillates within the range 31<N<47, and then the solution starts deteriorating rapidly for $N \geq 47$ (shown up to N = 55, figures 8 and 9). On the contrary, non-uniform DQM converge up to N = 100 and then diverge gradually, whereas the FEM (all cases) solutions converge smoothly for all N within the solution domain (shown up to N = 104, figures 8 and 10), showing best result at N = 101. Hence, the efficiency of our (conventional and optimum FEM) results are apparent.

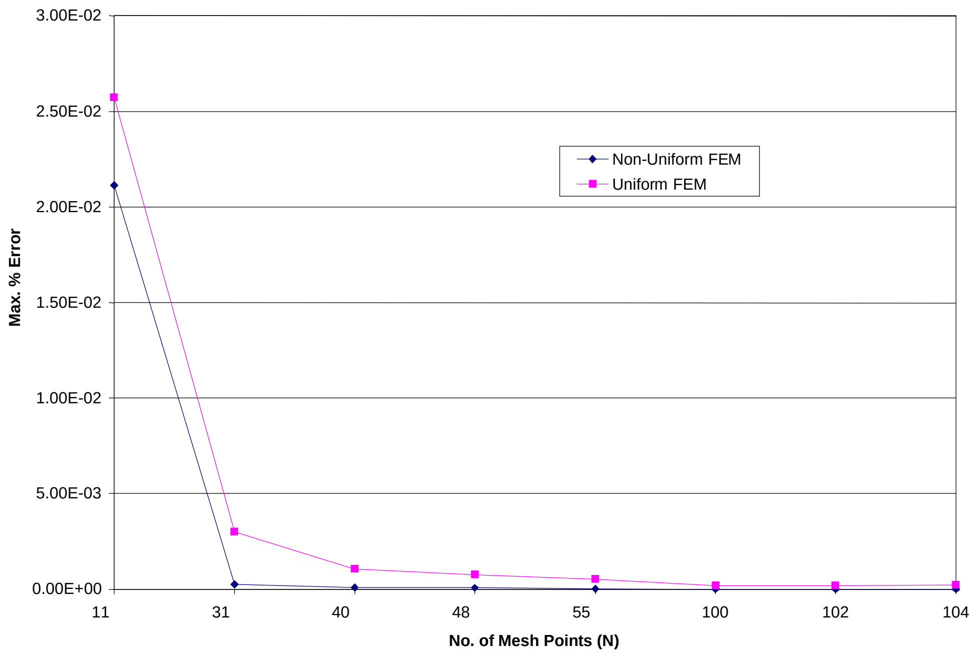

Figure 6. Comparison of convergence of fin-temperature in terms of maximum % error for conventional versus optimum FEM solution (N = 11-104)

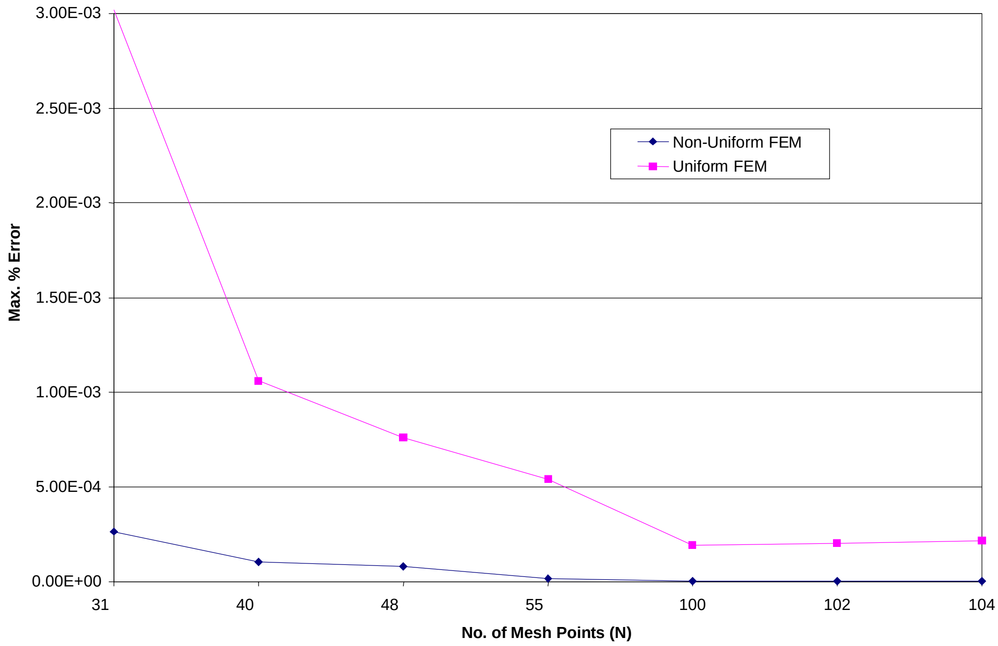

Figure 7. Comparison of convergence of fin-temperature in terms of maximum % error for conventional versus optimum FEM solution (N = 31-104)

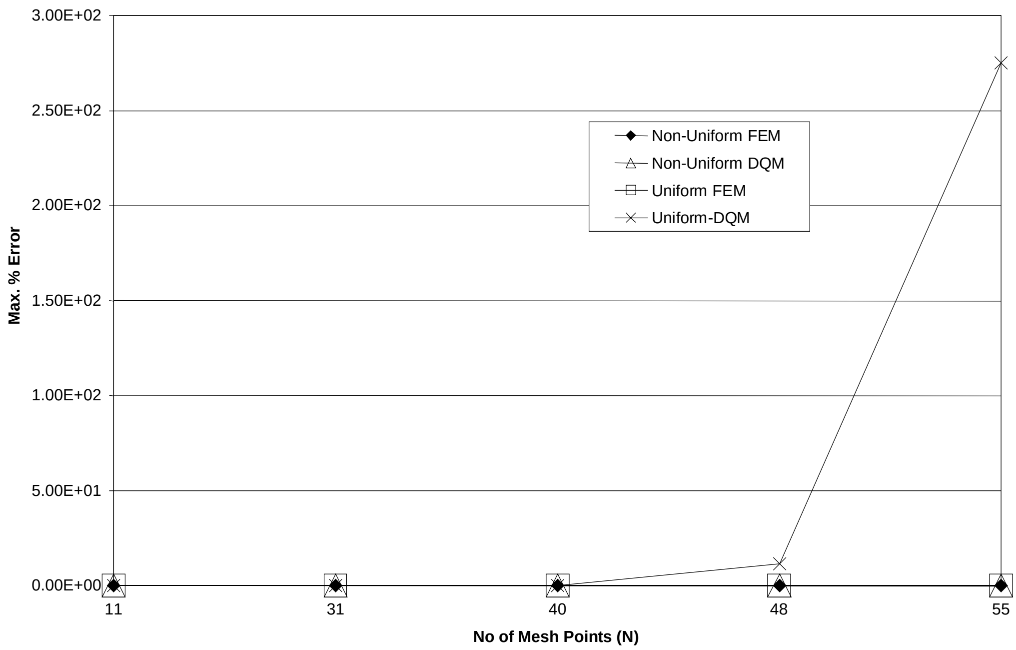

Figure 8. Comparison of convergence of fin-temperature in terms of maximum % error for all cases of FEM versus DQM solution (N = 11-55)

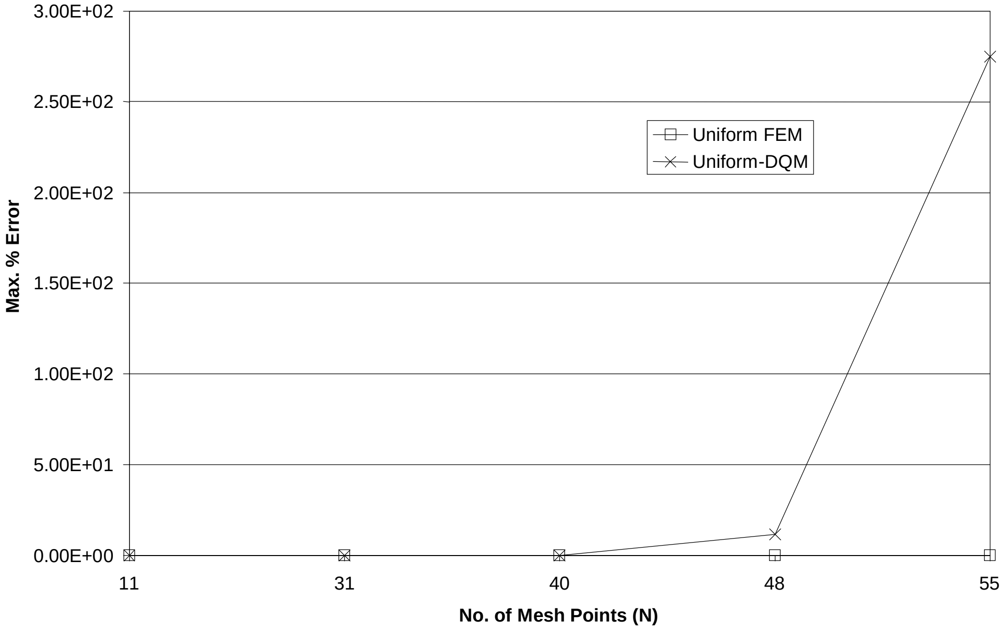

Figure 9. Comparison of convergence of fin-temperature in terms of maximum % error for conventional FEM versus DQM solution (N = 11-55)

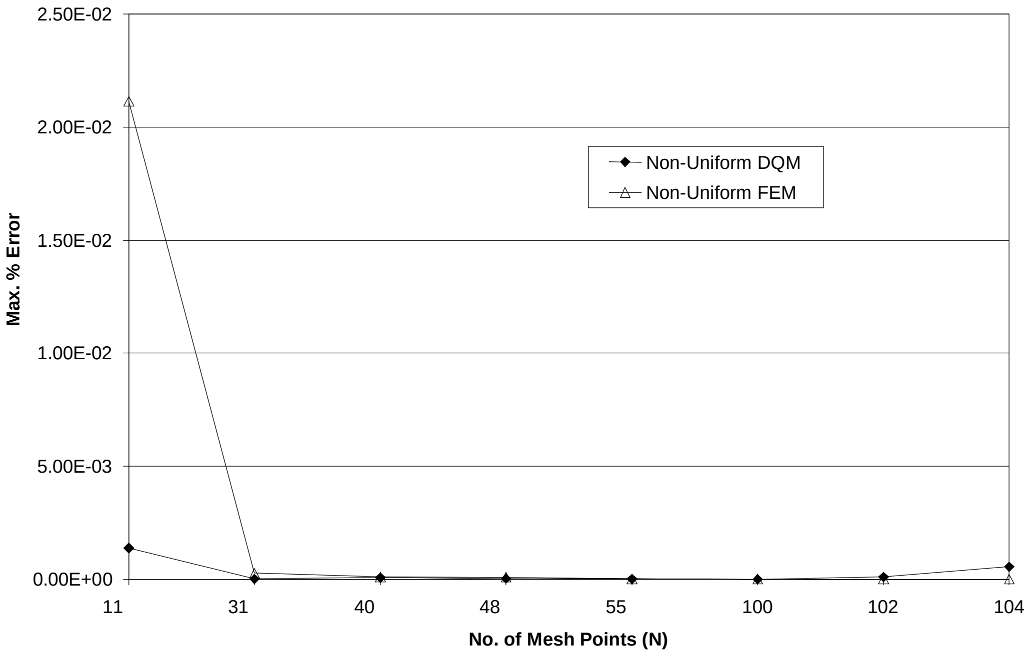

Figure 10. Comparison of convergence of fin-temperature in terms of maximum % error for optimum FEM versus DQM solution (N = 11-104).

5. Conclusions
The solutions of the temperature distribution in insulated-tip 1-D rectangular thin fin are computed numerically using FEM. The results are found to be in good agreement with the exact solution. It is found that the unequally spaced element distribution yield more accurate results than equally spaced for FEM solution. The solution converges smoothly as the number of elements reach to the optimum value. The results of EFEM shows outstanding improvement compared to CFEM and agree very well with EDQM with very minor variance showing its potentiality. Hence EFEM is suitable to test the temperature distribution scenario in any thin metal fin/plate prior to its design and practical implementation.

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